

# Two dimensional nonlinear dynamics of evanescent-wave guided atoms in hollow fiber

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(December 9, 1999)

## Abstract

We describe the classical and quantum two dimensional nonlinear dynamics of large blue-detuned evanescent-wave guiding cold atoms in hollow fiber. We show that chaotic dynamics exists for classic dynamics, when the intensity of the beam is periodically modulated. The two dimensional distributions of atoms in  $(x, y)$  plane are simulated. We show that the atoms will accumulate on several annular regions when the system enters a regime of global chaos. Our simulation shows that, when the atomic flux is very small, a similar distribution will be obtained if we detect the atomic distribution once each the modulation period and integrate the signals.

For quantum dynamics, quantum collapses and revivals appear . For periodically modulated optical potential, the variance of atomic position will be suppressed compared to the no modulation case. The atomic angular momentum will influence the evolution of wave function in two dimensional quantum system of hollow fiber.

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## I. INTRODUCTION

Evanescent light offers a very effective means of atom manipulation because of the strong induced dipole interaction between atoms and evanescent field when the field is far from resonance. Blue-detuned evanescent light, that is light tuned above the atomic resonance, can be used to guide atoms in a hollow fiber [1], [2], [3]. With blue-detuning, atoms are repelled from the high intensity field region near the fiber wall. The intensity in the evanescent field is significant over a distance of approximately  $\sim \lambda$  in the hollow region. If the inner diameter is several microns, the motion of atoms will be influenced by the evanescent field over a large transverse area inside the fiber. This system is very suited to studying nonlinear quantum and classical dynamics with two degrees of freedom in the transverse plane of the fiber. The maximum confined transverse atomic velocities are only several centimeters per second, therefore the atomic dynamics will be governed by the Schrödinger equation with an electric dipole coupling to the evanescent field. There are few experimental studies of quantum nonlinear dynamics for systems with two degrees of freedom and most of our understanding of quantum nonlinear dynamics and quantum chaos is based on simpler one degree of freedom systems, such as an atom in a modulated standing wave [4], or the microwave ionisation of hydrogen [5]. A notable example of experimental study of quantum chaos in two degree of freedom are the recent experiments on chaotic mesoscopic billiards [6,7]. It is clear that quite new phenomenon can appear in two degree of freedom systems [8] that are absent in one degree of freedom systems. Given the success of atom optics in providing tests of quantum chaos in one dimension it is advisable to consider what might be achieved using similar techniques for two dimensional systems. In this paper we will consider the two dimensional quantum and classical transverse motion of cold atoms in a hollow fiber with a periodically modulated evanescent field.

## II. THE OPTICAL POTENTIAL AND HAMILTONIAN

A two-level atom interacting with far-off-resonant inhomogeneous laser field has an effective potential [10] of the form

$$U(\mathbf{r}) = \frac{\hbar\Delta}{2} \ln(1 + p), \quad (1)$$

where  $\Delta$  is the detuning and  $p = \frac{\Omega^2/2}{\Delta^2 + \Gamma^2/4}$  is a saturation parameter, with the Rabi frequency  $\Omega$ . For far-off-resonant donut beams,  $p \ll 1$ , and thus

$$U(\mathbf{r}) = \frac{\hbar\Omega(\mathbf{r})^2}{4\Delta}. \quad (2)$$

In experiments [1], [2], [3] this potential has been used to guide cold atoms by reflecting them from evanescent light fields on the surface of the glass. For light striking the glass-vacuum interface at angle  $\theta$ , the evanescent intensity profile is

$$I(r) = I(0)\alpha^2 \exp[2\kappa(r - r_1)], \quad (3)$$

where  $I(0)$  is the input laser intensity at the fiber entrance,  $r_1$  is the inner radius of fiber, and the factors  $\alpha$  and  $\kappa$  are given in terms of the index of refraction  $n$ , inner reflection angle  $\theta$  and laser wavelength  $\lambda$  by  $\alpha = 2\sqrt{n^2/(n^2 - 1)} \cos \theta$ , and  $\kappa = (2\pi/\lambda)\sqrt{(n^2 \sin^2 \theta - 1)}$ . The Hamiltonian in transverse  $(x, y)$  plane for the system is

$$H_0 = \frac{p_x^2 + p_y^2}{2M} + K \exp[2\kappa(r - r_1)], \quad (4)$$

where  $K = \frac{\hbar\Gamma^2}{8\Delta} \frac{I(0)}{I_s} \alpha^2$  and  $I_s$  is the saturation intensity.

When the laser intensity is periodically modulated,  $I(0)$  becomes time dependent,  $I(0)[1 + \epsilon \cos(\omega t)]$  and the dynamics of the atom can be chaotic for certain initial conditions. There are two factors to consider in treating the two dimensional classic chaotic dynamics for evanescent wave guided atoms in a hollow fiber. Firstly, the boundary of the hollow fiber will limit the divergence of the atomic motion in the radial direction making it is easier to simulate the two-dimensional the dynamics of the atomic system (and also easier to probe the two dimensional distribution in experiment). Secondly, due to the small inner

radius of a hollow fiber, atoms with a small transverse velocity will be selected at the entry to the hollow fiber and it is not necessary to pre-cool the transverse temperature.

The effective optical potential depends on the inner radius  $r_1$  and decay coefficient  $\kappa$ . If the  $r_1$  is much larger than wavelength  $\lambda$  or  $\theta$  is too large, the potential will be very steep on the boundary and decay rapidly away from the inner surface of the fiber. In order to observe two dimensional chaotic dynamics in  $(x, y)$  plane,  $r_1$  should be the same magnitude as  $1/\kappa$ , so that the potential will decay slowly away from the surface of the hollow fiber. It is possible to make the reflection angle approach the critical reflection angle  $\sin^{-1}(1/n)$  by using several micro hollow fibers to guide the atoms and polish the incoming end of the hollow fiber at a specific acute angle [3].

We take  $r_1 = 2\mu m$  and  $\theta = 45^\circ$  and consider helium as an example. The parameters for helium are, line width  $\Gamma/2\pi = 1.6 MHz$ , mass  $M = 4m_p$ , wave length  $\lambda = 1.083\mu m$ , saturation intensity  $I_s = 0.16 mW/cm^2$  and recoil velocity  $v_r = 9 cm/s$ . We now define dimensionless parameters  $(\tilde{x}, \tilde{y}) = (2\kappa x, 2\kappa y)$ ,  $\tilde{r} = 2\kappa r$ ,  $(\tilde{p}_x, \tilde{p}_y) = (\frac{2\kappa p_x}{M\omega_0}, \frac{2\kappa p_y}{M\omega_0})$ , and  $\tilde{H} = H \frac{(2\kappa)^2}{M\omega_0^2}$ ,  $\tilde{\omega} = \omega/\omega_0$  and  $\tilde{t} = \omega_0 t$ , where  $\omega_0$  is a reference frequency. Omitting the tildes and defining  $\xi = \frac{K(2\kappa)^2}{M\omega_0^2}$ , the Hamiltonian can be rewritten as

$$H(t) = \frac{p_x^2 + p_y^2}{2} + \xi e^{\sqrt{x^2 + y^2} - r_1} (1 + \epsilon \cos \omega t), \quad (5)$$

with the canonical communication relations

$$[q_j, p_k] = i\hbar \delta_{jk}, \quad (6)$$

where  $q_j, p_k$  represents  $x, y$  and  $\hbar = \frac{\hbar(2\kappa)^2}{M\omega_0^2}$ , plays the role of a dimensionless Planck constant. If the dimensionless Plank constant  $\hbar$  approaches 1, the atomic motion in small hollow fibers will be correctly described by quantum dynamics.

### III. CLASSIC CHAOTIC DYNAMICS AND 2D DISTRIBUTION

Although the flux of guided atoms entering the fiber is very small, we still have the possibility to observe classic chaotic dynamics by integrating the signal over many periods

of the modulation (fig. 1). The integration length can be taken long enough to ensure that it includes large numbers of atoms. We will discuss the spatial distribution of the atom when the classical motion is chaotic. Using Hamilton's equations we find that the motion in the transverse plane without modulation ( $\epsilon = 0$ ) is described by the equations,

$$\dot{p}_x = -\xi \frac{x}{\sqrt{x^2 + y^2}} e^{\sqrt{x^2 + y^2} - r_1}, \quad (7)$$

$$\dot{p}_y = -\xi \frac{y}{\sqrt{x^2 + y^2}} e^{\sqrt{x^2 + y^2} - r_1} \quad (8)$$

$$\dot{x} = p_x, \quad (9)$$

$$\dot{y} = p_y. \quad (10)$$

Clearly there is only one stable fixed point on the axis ( $r = 0$ ).

The choice of modulation frequency  $\omega$  depends on the frequency of unperturbed periodic motion. For simplicity we assume  $y = 0$  and  $p_y = 0$ , so the expression for  $H$  simplifies to a one dimensional Hamiltonian system. The period of motion for the unperturbed Hamiltonian  $H_0$  is [11]

$$T_0 = \oint \frac{dx}{\partial H_0 / \partial p_x} = 2 \int_{-x_M}^{x_M} \frac{dx}{\sqrt{2(H_0 - \xi e^{\sqrt{x^2} - r_1})}}, \quad (11)$$

where  $x_M$  is determined by  $H_0 = \xi e^{\sqrt{x_M^2} - r_1}$ . Therefore

$$\omega_0 = \frac{\pi}{\int_{-x_M}^{x_M} \{2[H_0 - \xi e^{\sqrt{x^2} - r_1}]\}^{-1/2} dx}. \quad (12)$$

The graph of  $\omega_0$  versus  $H_0$  and  $x_M$  versus  $H_0$  is shown in fig. 2. We can select the modulation frequency  $\omega$  to control the position of the fixed points. Here we take  $\omega_0 = 265 KHz$ , dimensionless modulation frequency  $\omega = 2$  and set  $\xi = 50$ . The new fixed points will appear in  $x \approx 0.41$  and  $x \approx 2.87$ .

We use a symplectic integration routine [12], [13] to solve the equations of motion so as to preserve the Poisson bracket relation  $\{x(t), p_x(t)\} = 1$ , and thus maintain the Hamiltonian

character of the motion. We plot the stroboscopic portrait of the system at multiples of the period of modulation,  $t = (2\pi/\omega)s$ , where  $s$  is an integer referred to as the strobe number. From fig. 3 we can see that regions of globally chaotic motion will arise when  $\epsilon$  is large, together with some regular regions. A broad initial phase space distribution of atoms will enable some atoms to become trapped in these stable regions.

Laser cooling and trapping techniques have the ability to cool the atom to very low velocities and trap them with well localized momentum, however the position distribution is not so well localized. Therefore the appropriate description of the initial conditions is in terms of a probability density on phase space  $(x, y, p_x, p_y)$ . We define a classical state to be a probability measure on phase space of the form  $Q(x, y, p_x, p_y)dx dy dp_x dp_y$ . The probability density satisfies the Liouville equation

$$\frac{\partial Q}{\partial t} = \{H, Q\}_{q_i, p_i}, \quad (13)$$

where  $\{, \}_{q_i, p_i}$  is the Poisson bracket. This equation can be solved by the method of characteristics. To simulate an experiment, we assume atoms are initially randomly uniformly distributed on  $x^2 + y^2 < r_1^2$ ,

The momentum distributions for  $p_x$  and  $p_y$  are assumed to be Gaussian distributions. Therefore

$$Q_0(x, y, p_x, p_y) = Q_0(x)Q_0(y)Q_0(p_x)Q_0(p_y), \quad (14)$$

where

$$Q_0(p_i) = \frac{1}{2\pi\sigma_{p_i}} \exp [-(p_i - p_i(0))^2/2\sigma_{p_i}]. \quad (15)$$

The variances of  $p_x$  and  $p_y$  are related to the temperature  $T_i$

$$\sigma_{p_i} = k_B T_i / [M\omega_s^2 / (2\kappa)^2]. \quad (16)$$

We simulated the atomic system of  $10^4$  numbers and take  $\sigma_{p_i} = 0.1$ , which corresponds to radial rms velocity  $2cm/s$  for helium if  $\theta = 45^\circ$  and  $\omega = 2$ . The variance of  $Q$  function with time is

$$Q(\mathbf{r}, \mathbf{p}, \mathbf{t}) = \mathbf{Q}_0[\bar{\mathbf{r}}(\mathbf{r}, \mathbf{p}, -\mathbf{t}), \bar{\mathbf{p}}(\mathbf{r}, \mathbf{p}, -\mathbf{t})] \quad (17)$$

In the case of no modulation, atoms will accumulate around the fix point  $x = y = 0$ . When the modulation is added the atoms will diffuse in regions of chaotic motion but some will accumulate around several rings corresponding to fixed points at non-zero radius (fig. 4).

Because the inner size of hollow fiber is very small. The flux of guiding atoms entering the fiber will be low. But we can use a large blue-detuned shield laser, with the duration period of square wave equals to the modulation period  $T = 2\pi/\omega$ , and the time interval  $dT \ll T$ . When atoms emit out from the hollow fiber, they will travel freely without the optical potential. If the shield laser turn on, it will block atoms enter the detection region. In the short time interval  $dT$ , atoms will enter the detection region and the detector records the atomic distribution of momenta and positions. This procedure can be repeated as long as possible till we have integrated enough atomic numbers for many snap shots. The simulation shows that for integration of atomic numbers at different integer strobe numbers, it will produce similar results (fig. 5).

Because rings (fig. 4) represent new fixed points in phase space and the atomic radial momenta are almost approach zero (fig. 3). When atoms emit out and travel freely the shape of rings will not change very much in detection region. But the atomic velocity orientations outside the fixed points are randomly distributed. In order to measure the rings and small spatial distribution of atoms, high precision position measurements are required. The Raman-induced resonance imaging method [18], [19], [20] can be used to measure the positions with nanometer spatial resolution limited by the quantum uncertainty principle.

#### IV. TWO DIMENSIONAL QUANTUM NONLINEAR DYNAMICS

If the transverse temperature of atoms in hollow fiber is very cold, quantum nonlinear dynamics will result. Here we take the same parameters and same definition for dimensionless parameters as the classical dynamics and the dimensionless Plank constant  $\frac{\hbar(2\kappa)^2}{M\omega_s} \simeq 1$ .

The dimensionless 2D Schrödinger equation for atoms in hollow fiber

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \hat{H}(t) \psi(\mathbf{r}, t), \quad (18)$$

where

$$\hat{H}(t) = -\frac{\hbar^2}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(x, y, t). \quad (19)$$

and

$$V(x, y, t) = \xi e^{\sqrt{x^2+y^2}-r_1} (1 + \epsilon \cos \omega t), \quad (20)$$

where  $x^2 + y^2 \leq r_1^2$ .

For simplicity we assume the boundary condition for eq. (18) is

$$\psi(x, y, t)|_{\sqrt{x^2+y^2}=r_1} = 0. \quad (21)$$

The Split Operator Method [15] and 2D FFT (Fast Fourier Transformation) [17] will be used to obtain the numerical solution of Schrödinger eq. 18. The scheme in which the kinetic operator and potential operator are used to propagate the wave function separately:

$$\exp[-i\hat{\mathbf{H}}\delta t/\hbar] \sim \exp[-i(\hat{\mathbf{P}})^2\delta t/4\hbar] \exp[-i(\hat{\mathbf{V}})\delta t/2\hbar] \exp[-i(\hat{\mathbf{P}})^2\delta t/4\hbar], \quad (22)$$

and the computing errors are of  $O(\delta t^3)$ .

We assume at  $t = 0$  the wave function is a minimum uncertainty wave function. The initial variance of  $x, y$  are the same  $\sigma_x = \sigma_y = \sigma$ . The expression for this wave function is

$$\psi(x, y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-x_0)^2}{4\sigma}} e^{-\frac{(y-y_0)^2}{4\sigma}} e^{iP_{0x}x/\hbar} e^{iP_{0y}y/\hbar}. \quad (23)$$

In order to observe genuine quantum nonlinear behavior we have to wait for a time that is longer than the classical period. The time scale for quantum nonlinear dynamics  $T_{rev}$  is given by expression [9]

$$T_{rev} = T_0 \left( \frac{\hbar}{2} |\partial \omega_0 / \partial \bar{E}| \right)^{-1}. \quad (24)$$



In fig. 6 we have plotted the momentum mean  $\langle p_x \rangle$  and its variance as a function of time for  $\epsilon = 0$ . Quantum collapses and revivals appear for quantum nonlinear dynamics as expected.

In order to understand the influence of the modulation of the potential, we have plotted the variances of  $\langle x^2 \rangle - \langle x \rangle^2$ ,  $\langle p_x^2 \rangle - \langle p_x \rangle^2$  versus strobe numbers. In fig. 7 we found at integer strobe numbers, the modulation will increase the variance of momentum, but suppress the variance of position. The fluctuation of variances of positions and momentum are consistent with the Heisenberg uncertainty relationship in this quantum system.

If we rewrite  $\hat{H}$  in polar coordinator( $r, \theta$ ), we get

$$\hat{H} = -\frac{\hbar^2}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + V(r, t), \quad (25)$$

The general solution of eq.(19) will be [16]

$$\psi(r, \theta, t) = \sum_m a_m(r, t) e^{im\phi}, \quad (26)$$

where  $m$  is angular momentum quantum number. It shows that the evolution of wave function will be related to atomic angular momentum. In general for  $t > 0$  the probability  $|\psi(x, y)|^2$  will not be symmetric except the initial wave function has no angular momentum. In fig. 8 for initial conditions  $x_0 = y_0 = 0$  and  $p_{0x} = p_{0y} = 0$  we have plotted the probability at  $y = 0$  plane as a function of  $x$  at the time of strobe number 50, it shows that it will stay symmetric both for  $\epsilon = 0$  and 0.7. In fig. 9, for initial condition  $x_0 = y_0 = 0$  and  $p_{0x} = p_{0y} = 1$ , the probability distribution for  $t > 0$  will not be symmetric because initial atomic wave functions include angular momenta, and the interaction between atomic momenta and optical potential will destroy the spatial symmetry of probability. Therefore the atomic angular momentum plays an important rule in the evolution of wave function.

## V. CONCLUSION AND DISCUSSION

The atomic motion in small diameter hollow fiber is quantum mechanical if dimensionless Plank constant  $\hbar$  approach 1 . Quantum collapses and revivals will appear in the dynamics of the mean values . If the intensity is periodically modulated, we found the modulation will increase the variance of momentum , but suppress the variance of position at integer strobe numbers. The fluctuation of variances of positions and momentum is consistent with the Heisenberg uncertainty relationship in this system. In this two degree of freedom quantum system, the atomic angular momentum will influence the evolution of wave function.

Although the flux of guiding atoms entering the fiber is low, it's still possible to observe the classic chaotic dynamics by integrating the signals at different strobe numbers in experiment. The integration length can be taken long enough to ensure that it includes a large numbers of atoms. We have shown that an atom moving in an intensity modulated evanescent-wave field in hollow fiber can exhibit chaotic dynamics in the transverse plane. For atomic momenta  $p_x, p_y$  with Gaussian distributions, some atoms will become trapped in rings corresponding to radial fixed points of the modulated system in the moments of integer strobe numbers.

For the atomic average radial velocity is around recoil velocity and the spatial distribution size no larger than inner diameter of fiber, high resolution position and velocity measurements must be used. The Raman-induced resonance imaging method can reach the nanometer spatial resolution limited by the uncertainty principle. For atomic momentum distribution can be measured by Time of Flight absorption imaging method [22], [23] which is used to measure the temperature of super cold atoms in BEC experiment, or atomic velocity selection method using stimulated Raman transitions [21].

It is quite interesting that both classical chaotic dynamics and quantum dynamics have the possibility to be realized in experiment for atoms propagating in hollow fiber. Because the fiber's inner size is very small, it will select atoms with very small radial velocity to enter the fiber and there is no need to pre-cool the radial motion. In addition the fiber's

boundary will limit the divergence of radial motion, making it is easier to observe both two dimensional classical and quantum dynamics in experiment. We believe this is a far more practical scheme for observing the classical and quantum nonlinear dynamics of radially confined atoms than other schemes such as those that use a donut beam scheme [14].

## **VI. ACKNOWLEDGMENT**

One of authors(XML) would like to thank Dr. Kenneth Baldwin at Australia National University for useful discussion about physical parameters of helium and hollow fiber.

## REFERENCES

- [1] H. Ito, K. Sakaki, W. Jhe, M. Ohtsu, Opt. Commun. 141, 43-47(1997).
- [2] H. Ito, T. Nakata, K. Sakaki, M. Ohtsu, Phys. Rev. Lett. 76, 4500(1996).
- [3] Michael J. Renn, Elizabeth A. Donley, Eric Cornell, Carl E. Wieman, Dana Z. Anderson, Phys. Rev. A. 53, R648(1996).
- [4] F.L.Moore, J.C.Robinson, C. Bharucha, P.E.Williams and M.G.Raizen, Phys. Rev. Lett, **73**, 2974 (1994).
- [5] R.V.Jensen,S.M.Susskind and M.M.Sanders, Phys. Rep. **201**, 1 (1991).
- [6] C.M.MArcus,A.J.Rimberg,R.M.Westervelt,P.F.Hopkins, and A.C.Gossard, Phys. REv. Lett. **69**, 506 (1992).
- [7] R. P. Taylor,R. Newbury,A. S. Sachrajda,Y. Feng,P. T. Coleridge,C. Dettmann,Ningjia Zhu,Hong Guo, A. Delage,P.J. Kelly, and Z. Wasilewski, Phys. Rev. Lett **78**, 1952 (1997).
- [8] A.R.Kovlosky, Phys. Rev. E **50**, 3569 (1994).
- [9] Wenyu Chen, S.Dyrting and G.J. Milburn, Australia Journal of Physics, 49, 777-818 (1996).
- [10] C. Cohen-Tannoudji, in *Fundamental Systems in Quantum Optics*, Proceedings of the Les Houches Summer School(North-Holland, Amsterdam, 1992).
- [11] A.J.Lichtenberg, M.A. Lieberman (1983), *Regular and Stochastic motion*, Springer-Verlag, New York ,Heidelberg Berlin.
- [12] Etienne Forest and Martin Berz. Canonical integration and analysis of periodic maps using non-standard analysis and Lie methods. In Kurt Bernardo Wolf, editor, *Lie Methods in Optics II* , page 47, Berlin Heidelberg, 1989, Springer-Verlag.

- [13] S.Dyrting. Ph.D thesis (1995), Department of Physics, University of Queensland.
- [14] X.M. Liu and G.J. Milburn, Phys. Rev. E, 59, 2842(1999).
- [15] Ronnie Kosloff, J. Phys. Chem, 92, 2087(1988).
- [16] Lothar Collatz (1986), *Differential Equations: An Introduction with Applications* , John Wiley & Sons.
- [17] William H Press et al (1992), *Numerical Recipes in C* , Cambridge, University Press.
- [18] J.E. Thomas, Phys. Rev. A. 42, 5652(1990).
- [19] K. D. Stokes, C.Schnurr, J. R. Garner, M. Marable, G. R. Welch, and J. E. Thomas, Phys. Rev. Lett. 67, 1997(1991).
- [20] J. R. Gardner, M. L. Marable, G. R. Welch, and J. E. Thomas, Phys. Rev. Lett. 70, 3404(1993).
- [21] Mark Kasevich, David S. Weiss, Erling Riis, Kathryn Moler, Steven Kasapi and Steven Chu, Phys. Rev. Lett. 66, 2297(1991).
- [22] K. B. Davis, M. -O. Mewes, M. R. Andrew, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969(1995).
- [23] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. 75, 1687(1995).

## FIGURES

FIG. 1. *The diagram of proposed experiment. The large blue-detuned shield laser will block atoms enter the detection region except in the short time interval  $dT \ll T$ . The integration of many snap shots will have the similar effect of atomic distribution compared to one snap shot of large atomic numbers.*

FIG. 2. *a. The relations between frequency of motion and Hamiltonian  $\omega_0 \sim H_0$  and  $x_M \sim H_0$ .*

FIG. 3. *Stroboscopic portrait of the system with  $\epsilon = 0.7$ ,  $p_x(0) = 0$ ,  $p_y = 0$  and  $y = 0$ . The maximum strobe number is 500.*

FIG. 4. *The atomic distribution in  $(x, y)$  plane at the strobe number 50 for  $\epsilon = 0.7$ . The  $10^4$  atoms were taken in phase space. The atoms were initially distributed on  $x^2 + y^2 \leq r_0^2$  region. The momenta of  $p_x, p_y$  are Gaussian distributions and  $\sigma_{p_x} = \sigma_{p_y} = 0.1$ .*

FIG. 5. *The atomic distributions of  $10^4$  numbers in  $(x, y)$  plane. After strobe number 50, we count 200 atoms at the time of every integer strobe number until  $s = 150$ .*

FIG. 6. *The average momentum  $\langle p_x \rangle$  and momentum variance  $\langle p_x^2 \rangle - \langle p_x \rangle^2$  evolutions with time. The initial wave function is the least uncertainty state,  $\sigma_x = \sigma_y = 0.1$ . For (a),  $x_0 = y_0 = 0$ ,  $p_{0x} = p_{0y} = 1.0$ . For (b),  $x_0 = y_0 = 0.5$ ,  $p_{0x} = p_{0y} = 0.0$ .*

FIG. 7. *The position variance  $\langle x^2 \rangle - \langle x \rangle^2$  and momentum variance  $\langle p_x^2 \rangle - \langle p_x \rangle^2$  versus strobe numbers. The initial wave function is the least uncertainty state,  $\sigma_x = \sigma_y = 0.1$ . For (a) and (b)  $x_0 = y_0 = 0$ ,  $p_{0x} = p_{0y} = 1.0$ . For (c) and (d),  $x_0 = y_0 = 0.5$ ,  $p_{0x} = p_{0y} = 0.0$ .*

FIG. 8. *The probability distributions  $|\psi(x, 0)|^2$  for  $y = 0$ . The initial wave function is the least uncertainty state,  $\sigma_x = \sigma_y = 0.1$ ,  $x_0 = y_0 = 0$ ,  $p_{0x} = p_{0y} = 0$ . The point line is distribution for  $t = 0$ , the dashed line is distribution for  $t = 50T$ , no modulation  $\epsilon = 0$ , and the solid line is the distribution for  $t = 50T$ ,  $\epsilon = 0.7$ .*

FIG. 9. *The probability distributions  $|\psi(x,0)|^2$  for  $y = 0$ . The initial wave function is the least uncertainty state,  $\sigma_x = \sigma_y = 0.1$ ,  $x_0 = y_0 = 0$ ,  $p_{0x} = p_{0y} = 0.1$ . The point line is distribution for  $t = 0$ , the dashed line is distribution for  $t = 50T$ , no modulation  $\epsilon = 0$ , and the solid line is the distribution for  $t = 50T$ ,  $\epsilon = 0.7$ .*



















